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The Behavior of a Periodically-Forced Nonlinear System Subject to Additive Noise

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We continue the study of a nonlinear first-order dynamical system first considered by Chen. This model is characterized by a multiplicative periodic forcing term and additive dichotomous noise in place of the white noise of Chen's analysis. Two parameters are used to characterize the qualitative properties of such a system, the mean first-passage time to the ends of the interval and the Fourier spectrum generated by the solution of the equation. We show that the mean first-passage time is monotonic in the amplitude of the periodic force and exhibits a resonant dependence on its frequency. In addition the substitution of dichotomous for white noise leads to a systematic change in the ability to smooth out the peaks in the Fourier spectrum of the solution.

KEY WORDS: Stochastic resonance; first-passage times; periodic forces.

1. INTRODUCTION

Since the initial suggestion and analysis of the basic ideas behind the notion of stochastic resonance,[1,2] a considerable amount of analysis has been devoted to elucidating properties of dynamical systems subject simultaneously to both noise and periodic forcing functions. Such systems can exhibit a variety of behavior not accessible when only one type of driving force acts on the system. Recently Chen[3] has discussed some properties of a Brownian rotor whose equation of motion is

$$\dot{\theta} = v \sin(\omega t) \sin(\theta) + (2D)^{1/2} n(t) \tag{1}$$

where θ is an angle and $n(t)$ is white noise characterized by the second-order properties

$$\langle n(t) \rangle = 0, \qquad \langle n(t)\, n(s) \rangle = \delta(t-s) \tag{2}$$

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Since θ is an angular variable, the constants v and D have dimensions $(\text{time})^{-1}$. Chen investigated a property of the system behavior which he termed noise-induced instability. This is defined by the statement that the noise in Eq. (1) eventually drives the system through the two fixed endpoints of the interval which would not be the case in the absence of noise. Facets of erratic behavior are exhibited by both the system trajectory and the associated power spectrum.

The noise-free version of Eq. (1) can be solved exactly because the periodic term occurs multiplicatively rather than additively as in more usual analyses of nonlinear resonance. The solution to Eq. (1) with $D=0$ is readily shown to be

$$\theta(t)=2\tan^{-1}\left\{\tan\left(\frac{\theta_0}{2}\right)\exp\left[v\,\frac{1-\cos(\omega t)}{\omega}\right]\right\} \tag{3}$$

in which $\theta_0=\theta(0)$. It is evident from this expression that in a noise-free system $\theta(t)$ cannot reach the boundaries of the fundamental interval at $\pm\pi$ unless it is at one of them initially. However, in the presence of noise the system will exhibit diffusive behavior and, for any initial condition, will eventually reach one or the other of the endpoints. The competition between noise and the periodic coefficient has an interesting aspect, as remarked on by Chen,[3] namely that increasing the amplitude of the deterministic term also increases the importance of the noise term in Eq. (1). This seemingly paradoxical statement is easily explained by the fact that an increase in the periodic term causes $\theta(t)$ to approach the interval endpoints. When $\theta(t)$ is in the neighborhood of one of these points the effect of the random force is to cause $\theta(t)$ to reach one or the other value $\pm\pi$ in a finite time.

In the present paper we generalize the work of Chen by considering three further aspects of the behavior of the dynamical system in Eq. (1) to elucidate the interplay between noise and the periodicity that one finds in the solution of the noise-free equation. The first of these is contained in a study of the first-passage time problem associated with the system to reach one of the two points $\pm\pi$. Here we generalize a recent study of the same problem for the simpler linear case,[4] which is defined by

$$\dot{\theta}=v\sin(\omega t)+(2D)^{1/2}\,n(t) \tag{4}$$

The results of that investigation established resonant behavior of the mean first-passage time considered as a function of the frequency. We show that this, too, occurs in the nonlinear system in Eq. (1). We then consider some effects on the dynamical system of changing $n(t)$ to either a deterministic square wave signal or dichotomous noise, to determine which properties of the system are truly dictated by noise and which can be reproduced by a deterministic forcing function allowed to take on both positive and negative values.

2. THE FIRST-PASSAGE-TIME PROBLEM

As mentioned, $\theta(t)$ will always reach one of the endpoints of the fundamental interval when noise is present. One is then interested in properties of the first-passage time, and in particular, how this first-passage time is influenced by the competing influences of the periodic and noise terms.

We will express the Smoluchowski equation equivalent to Eq. (1) in terms of the dimensionless variables $y = \theta/\pi$, $\tau = Dt/\pi^2$, and $\rho = \pi^2\omega/D$:

$$\frac{\partial p}{\partial \tau} = \frac{\partial^2 p}{\partial y^2} - \varepsilon \sin(\rho\tau) \frac{\partial}{\partial y} [\sin(\pi y)\, p] \tag{5}$$

where $\varepsilon = \pi v/D$. The transformation from θ to y takes the interval $\pm\pi$ into ± 1. By convention we assume that $\varepsilon > 0$. We seek the solution to this equation that satisfies the initial and boundary conditions

$$p(y, 0) = \delta(y - y_0); \qquad p(-1, \tau) = p(1, \tau) = 0 \tag{6}$$

We proceed by expanding $p(y, \tau)$ in a Fourier sine series that ensures that the boundary conditions are satisfied. That is to say, we write

$$p(y, \tau) = \sum_{n=1}^{\infty} a_n(\tau) \sin(n\pi y) \tag{7}$$

When this series is substituted into Eq. (5) then, by taking advantage of the orthogonality of the sine functions, one reduces the problem of solving Eq. (5) to that of solving an infinite set of differential equations:

$$\dot{a}_n + n^2\pi^2 a_n = -\varepsilon_1 \sin(\rho\tau)\, n[a_{n+1} - a_{n-1}] \tag{8}$$

in which $\varepsilon_1 = \pi\varepsilon/2$. A knowledge of the $a_n(\tau)$ and the Laplace transforms of these functions $\{\hat{a}_n(s)\}$ allow us to express the survival probability $S(\tau \mid y_0)$ and the associated mean first passage $\langle \tau \mid y_0 \rangle$ as

$$S(\tau \mid y_0) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{a_{2n+1}(\tau)}{2n+1}, \qquad \langle \tau \mid y_0 \rangle = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\hat{a}_{2n+1}(0)}{2n+1} \tag{9}$$

Finding a solution to the full set of differential equations in Eq. (8) is not a trivial task because of the occurrence of the time-dependent term on the right-hand side. It is possible, however, to calculate successive approximations to the solution in the form of a perturbation series provided that $\varepsilon_1 \ll 1$. We then expand $a_n(\tau)$ in the series

$$a_n(\tau) = \sum_{j=0}^{\infty} a_n^{(j)}(\tau)\varepsilon_1^j \tag{10}$$

The solution is then generated from the hierarchy of equations

$$\dot{a}_n^{(0)} + \pi^2 n^2 a_n^{(0)} = 0 \tag{11a}$$

$$\dot{a}_n^{(j+1)} + \pi^2 n^2 a_n^{(j+1)} = -(\pi/2)\sin(\rho\tau)\, n[a_{n+1}^{(j)} - a_{n-l}^{(j)}] \tag{11b}$$

In order to find the mean first-passage time. it is expedient to generate the corresponding recursion relation for the Laplace transforms. Because of the appearance of the sinusoidal term in Eq. (8) this is

$$\hat{a}_n^{(0)}(s) = \frac{2\sin(n\pi y_0)}{s + \pi^2 n^2} \tag{12a}$$

$$\begin{aligned}\hat{a}_n^{(j+1)}(s) = \frac{in}{2(s+\pi^2 n^2)} \{ & [a_{n+1}^{(j)}(s - i\rho) - a_{n+1}^{(j)}(s + i\rho)] \\ & - [a_{n-1}^{(j)}(s - i\rho) - a_{n-1}^{(j)}(s + i\rho)]\}\end{aligned} \tag{12b}$$

The inversion of Eq. (12a) is, of course, trivial. From that inversion and the recursion relation in Eq. (12b) we can generate approximations to $\langle \tau \mid y_0 \rangle$, which, to terms of order in ε_1 is

$$\begin{aligned}\langle \tau \mid y_0 \rangle \sim & \frac{1}{2} y_0(1 - y_0) \\ & + \frac{\varepsilon_1}{\pi^2}\rho \sum_{n=0}^{\infty} \frac{1}{2n+1} \left\{ \frac{\sin[2(n+1)\pi y_0]}{\rho^2 + 16\pi^4(n+1)^4} - \frac{\sin 2n\pi y_0}{\rho^2 + 16\pi^4 n^4} \right\}\end{aligned} \tag{13}$$

It is clear that the term proportional to ε_1 has an extremum when considered as a function of ρ. To see this, we can, to a good approximation, replace the indicated sum by the $n = 0$ term along, which allows us to replace Eq. (13) by the simplified expression

$$\langle \tau \mid y_0 \rangle \sim \frac{1}{2} y_0(1 - y_0) + \frac{\varepsilon_1}{\pi^2}\sin(2\pi y_0)\frac{\rho}{\rho^2 + 16\pi^4} \tag{14}$$

from which the resonant behavior is evident, although it is small because of our assumption that $\varepsilon_1 \ll 1$. Notice that because of the sine term on the right-hand side the extremum can be either a minimum or a maximum, depending on the initial position y_0. The approximation in this last equation implies that when $y_0 < 1/2$ the perturbation term leads to an increase of $\langle \tau | y_0 \rangle$ with respect to the zeroth-order term, while when $y_0 > 1/2$ the mean first-passage time is decreased. This can be understood physically, since when y_0 is small the field tends first to move the particle in the direction of the trap at $y = 1$. After a cycle of motion biased in this way untrapped particles tend to reverse direction, thereby prolonging their average sojourn time and preserving their status as untrapped particles. On the other hand, particles closer to the trap at $y = 1$ move toward the trap there, with the result that a greater number of them are trapped at early times. An important further observation is that in the present example the competition between noise and the effect of an oscillatory field depends not only on the amplitudes but also on the forcing frequency.

3. A PERIODIC DETERMINISTIC FIELD

Chen considered a problem which essentially compared the effects of the inherent nonlinearity to those due to noise. The balance of these is the source of this so-called erratic behavior.[3] We here discuss the behavior of a system in which the noise term is replaced by a deterministic telegraphic signal so that the equation governing the dynamics is

$$\dot{y} = \varepsilon \sin(\tau) \sin(y) + Ah(\tau) \tag{15}$$

in which A is a constant and the function $h(\tau)$ is defined by

$$h(\tau) = \begin{cases} +1 & \text{for} \quad \tau \in [2m\Omega, (2m+1)\Omega] \\ -1 & \text{for} \quad \tau \in [(2m+1)\Omega, (2m+2)\Omega] \end{cases} \tag{16}$$

in which $m = 0, 1, 2,...$. Our investigation is aimed at discovering whether the system defined by Eqs. (15) and (16) exhibits any symptoms in the solution similar to Chen's erratic behavior. Of course one cannot expect true erratic behavior in Chen's sense of the term since there is no random element in our defining equation. All of the conclusions discussed below are based on results of numerical solutions to Eq. (15).

We have chosen the time variable in the definition of Eq. (15) so that the frequency in the deterministic term is effectively equal to 1. Our definition of the function $h(\tau)$ defines a second frequency Ω. We might expect $y(\tau)$ to be a periodic (although possibly quite complicated) function of τ when Ω is a rational number and a quasiperiodic function when it is irrational.

A qualitative picture of some properties of the solution appear in terms of its power spectrum defined in terms of the Fourier transform of $y(t)$ as $g(\eta) = |\mathscr{F}\{y(t)\}|^2$, where η is the transform parameter. When ε is small one expects the periodicity in $h(\tau)$ to be the significant factor that determines the resonant peaks in $g(\eta)$, while for larger ε both frequencies, that coming from the sinusoidal term in Eq. (1) and that arising from the periodicity in $h(\tau)$. In the following section we will consider the effects of dichotomous noise, in which case we want to establish a correspondence between the amplitude A and the diffusion constant D, which is equal to 0.001 in Chen's article. In this deterministic case this will be done by taking the average of A^2 over a complete cycle of duration $2\pi/\Omega$ and equating it to $2D$. This yields the value $A = 0.04472$.

When ε is sufficiently small, peaks in the power spectrum will principally be determined by the telegraph signal given in Eq. (17). We found that with $\Omega = 0.5$ for ε in the range 0–1.5, peaks of the spectrum considered as a function of η occurred at multiples of 0.5. As ε increases, the influence of the nonlinear term in Eq. (15) becomes more significant. Then it is found that the peaks at $\eta = 1, 2,...$ disappear, leaving only those at η values at the values $n + 1/2$, where $n = 0, 1, 2,...$. Our numerical calculation indicates that for the values $\Omega = 1/4$ and $\varepsilon = 0.1$ an increase in A from 0.5 to 10 converts the minima of $g(\eta)$ at the values $\eta = 1, 2$, and 3 into maxima at those values. The same inversion occurs if one fixes $A = 0.5$ and increases ε from 0.1 to 3. We mention in passing that for the irrational value $\Omega = \sqrt{2}$ we found the transform $g(\eta)$ to be practically flat for all values of ε, which suggests that a more precise definition of the terminology "erratic behavior" would be of some value in distinguishing that from chaotic or a purely periodic signal.

4. DICHOTOMOUS NOISE

The work of Chen was directed at the study of the effects of white noise on properties of the solution to Eq. (1). Here we consider a particular case of colored noise, and more specifically, dichotomous noise. That is, $n(t) = \pm A$, the durations of time spent in a single sojourn in either the positive or negative states, are taken as random variables, the transitions occurring at rate Ω. We examine the behavior of the normalized function $g(\eta)/g(0)$ as a function of η at different values of the amplitude ε when Ω is held fixed. In Fig. 1 we plot $g(\eta)/g(0)$ against η for the values $A = 0.04472$ and $\Omega = 2$ for the three values $\varepsilon = 1$, 2, and 3. When $\varepsilon = 1$ there is a dominant peak at $\eta = 1$ and much smaller ones (unobservable on the scale of the curve in Fig. 1a) at $\eta = 2$ and 3. As the amplitude ε increases, these peaks become less prominent and more or less disappear at $\varepsilon = 3$. This is similar to the case of white noise case analyzed by Chen, the results being sum-

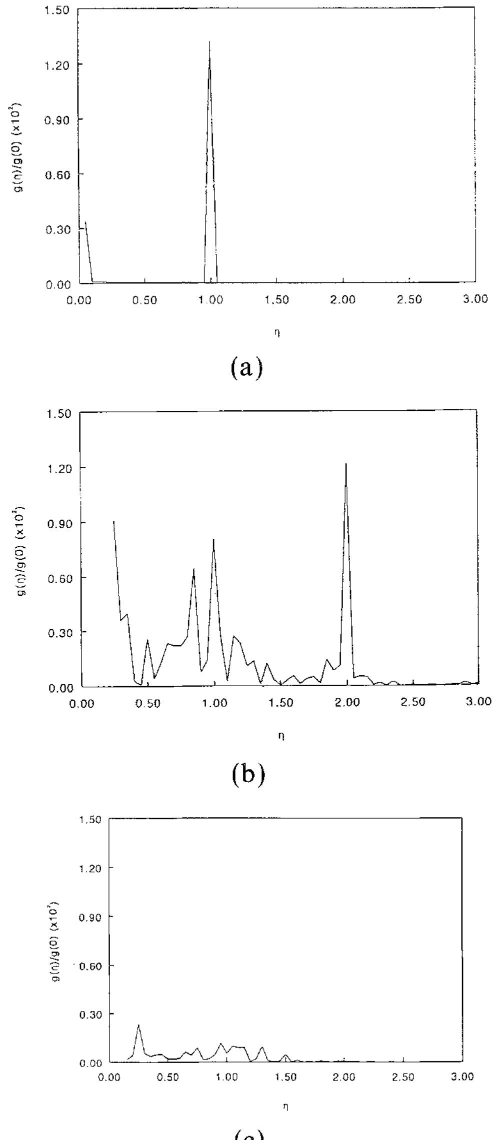

Fig. 1. (a) A plot of $g(\eta)/g(0)\times 10^2$ as a function of η for $\Omega=2$ and $\varepsilon=1$. The sharp peak at $\eta=1$ is evident. Much smaller peaks occur at $\eta=2$ and 3. These are too small to be seen on the scale of the graph. (b) The same plot for $\varepsilon=2$ on the same scale. The peak at $\eta=1$ is still in evidence, but the one at $\eta=2$ has become much more prominent, while that at $\eta=3$ is still too small to be seen. (c) The same plot for $\varepsilon=3$. None of the peaks show up in this figure.

marized in Fig. 2 of this paper. When the frequency is decreased to 1/2 the peaks in the spectrum disappear at lower values of ε. Our numerical results suggest that at the smaller values of Ω dichotomous noise smooths out peaks in the spectrum more effectively than white noise, while at the larger values of Ω the opposite conclusion holds.

5. DISCUSSION

Our results extend the analysis of Chen of a particular nonlinear equation which is interesting because it reflects the combined effects of noise and periodicity. In our analysis we looked both at the first-passage-time problem associated with the particle leaving the fundamental interval $(-\pi, +\pi)$ and at the spectrum generated by the solution of Eq. (1).

In our study of the first-passage-time problem we have found that significant changes in the mean first-passage time can be induced by changing either the amplitude or frequency of the periodic term, and indeed there is a "resonant" frequency at which the mean first-passage time is a minimum. This is in agreement with the results of Fletcher *et al.* for the ordinary diffusion process.[3]

A numerical solution of the equation analogous to Eq. (1) except for having the noise term replaced by a symmetric pulse shows that the Fourier spectrum has distinct peaks when the two frequencies appearing in the time-dependent terms are commensurate. When, however, the frequencies are incommensurate, the Fourier spectrum is flattened to a considerable degree, which is similar to the "erractic" spectrum found in Chen's analysis. This raises the question of whether, and under what circumstances, the concept of erratic behavior can be expected to be a useful one. In particular one might question its use for systems not having enough degrees of freedom to manifest deterministic chaos.

Our analysis of Eq. (1) with dichotomous rather than white noise showed that the solutions are similar in that they both can be described as being diffusive. At small switching frequencies of the dichotomous noise the peaks in the Fourier spectrum of $x(t)$ are smoothed out more for small values of the amplitude than is the case for white noise of the same strength, while at higher frequencies the relation is reversed.

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